

The University of Chicago

THE VARIATIONS OF ABSORPTION-
LINE CONTOURS ACROSS THE
SOLAR DISC

A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF DOCTOR
OF PHILOSOPHY

DEPARTMENT OF ASTRONOMY AND ASTROPHYSICS

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ABSTRACT

In this paper a new method for calculating theoretical contours of absorption lines at various points on the solar disc is described, the general case in which the ratio of the line absorption coefficient to the continuous absorption coefficient is variable through the atmosphere being considered. The method requires the solution of the appropriate boundary-value problem of line formation in approximations higher than those hitherto considered. Various tables which are needed for the practical construction of line contours are provided.

Theoretical line contours are constructed on the "third approximation," which provides predictions for three points on the solar disc, representing the fractions 0.97, 0.75, and 0.36 of the solar radius from the center. From the comparison between theoretical and observed contours for certain selected lines, it is found that the general trends in the observed center-limb effects are accounted for.

1. *Introduction.*—The variations of line contours across the solar disc provide information concerning the structure of the solar atmosphere and the physical processes relevant to the formation of absorption lines. In most solutions to the theoretical problem predicting such variations, it has been customary to assume that the ratio η_ν of the line and continuous absorption coefficients is a constant through the atmosphere. While the approximate constancy of η_ν through model stellar atmospheres is met under certain special conditions, it is, nevertheless, an assumption which is hard to justify in general. It would, therefore, be useful to have a treatment of the problem which does not make such restrictive assumptions and which is, at the same time, sufficiently simple to enable applications to practical cases. It is the object of this paper to provide such a theory and show that it is feasible by constructing theoretical contours for some typical cases and comparing them with observations.

2. *A method for computing the theoretical contours of absorption lines.*—The equation of transfer appropriate for the problem of the formation of absorption lines, originating from a ground state, is, in a standard notation,¹

$$\mu \frac{dI_\nu}{dt_\nu} = I_\nu - \frac{(1-\epsilon) \eta_\nu}{2(1+\eta_\nu)} \int_{-1}^{+1} I_\nu d\mu - \frac{1+\epsilon\eta_\nu}{1+\eta_\nu} B_\nu(t_\nu). \quad (1)$$

By introducing the quantity

$$\lambda_\nu = \frac{1+\epsilon\eta_\nu}{1+\eta_\nu}, \quad (2)$$

equation (1) can be re-written in the form

$$\mu \frac{dI}{dt} = I - \frac{1}{2} (1-\lambda) \int_{-1}^{+1} I d\mu - \lambda B(t), \quad (3)$$

where for convenience we have suppressed the suffix ν . For the problem of predicting the equivalent widths of absorption lines, we can, without loss of generality, set $\epsilon = 0$; for, as is well known, a nonvanishing ϵ is required only for accounting for the central intensities. Following a general method, based on Gauss's formula for numerical quadratures,

¹ See, e.g., B. Strömberg, *Aph. J.*, **86**, 1, 1937.

which has recently been developed by Chandrasekhar² for the solution of the various problems of radiative transfer in the theory of stellar atmospheres, we replace equation (3) by the system of $2n$ linear equations

$$\mu_i \frac{dI_i}{dt} = I_i - \frac{1}{2} (1 - \lambda) \sum_{j=-n}^{+n} a_j I_j - \lambda B(t) \quad (i = \pm 1, \dots, \pm n) \quad (4)$$

in the n th approximation, where the μ_i 's are the zeros of the Legendre polynomial of order $2n$ and the a_j 's are the appropriate Gaussian weights.

In the "standard case" the ratio η of the line absorption coefficient to the coefficient of continuous absorption at the frequency ν of the absorption line is assumed to be independent of the optical depth t in the line. The quantity λ has then a constant value λ_0 (say), and the Planck intensity $B(t)$ is expressible as a linear function of t ,

$$B(t) = a_{\nu_0} + bt, \quad (5)$$

where $b = \lambda_0 b_{\nu_0}$. Expressions for a_{ν_0} and b_{ν_0} are obtained by expanding B as a Taylor series from $t = 0$, using Milne's relation³ for the temperature distribution, namely,

$$T^4 = T_0^4 (1 + \frac{3}{2} \tau), \quad (5')$$

where T_0 is the surface temperature and τ is the optical depth in the continuous spectrum. The solution⁴ to the system of equations (4) appropriate for this case is

$$I_i^{(0)}(t) = \sum_{a=1}^n \frac{L_a^{(0)} e^{-k_a t}}{1 + \mu_i k_a} + (\mu_i b + a_{\nu_0} + bt) \quad (i = \pm 1, \dots, \pm n), \quad (6)$$

where the k_a 's are the n positive roots of the characteristic equation

$$1 = (1 - \lambda_0) \sum_{j=1}^n \frac{a_j}{1 - \mu_j^2 k^2} \quad (7)$$

and the $L_a^{(0)}$'s ($a = 1, \dots, n$) are the n constants of integration to be determined by the boundary conditions

$$\sum_{a=1}^n \frac{L_a^{(0)}}{1 - \mu_i k_a} + a_{\nu_0} = \mu_i b \quad (i = 1, \dots, n). \quad (8)$$

This solution has been obtained in its numerical form⁵ in the first three approximations. The roots k_a ($a = 1, \dots, n$) for various values of λ_0 are tabulated in paper IV (Table 1), and the constants $L_a = \lambda_0 L_a^{(0)}/b$ ($a = 1, 2, 3$) in the third approximation are given in paper VI (Table 1) for various values of λ_0 and

$$x = \frac{n}{u} \frac{e^u - 1}{e^u}, \quad (9)$$

where

$$u = \frac{h\nu_0}{kT_0} \quad \text{and} \quad n = \frac{\kappa_{\nu_0}}{\kappa}, \quad (10)$$

² *Ap. J.*, **100**, 76, 1944. This paper will be referred to hereafter as "II."

³ Cf. E. A. Milne, *Handb. d. Ap.*, **3**, 119, Berlin, 1929.

⁴ Cf. II, p. 86.

⁵ C. U. Cesco, S. Chandrasekhar, and J. Sahade, *Ap. J.*, **100**, 355, 1944, and *Ap. J.*, **101**, 320, 1945. These papers will be referred to hereafter as "IV" and "VI," respectively.

κ_{ν_0} being the continuous absorption coefficient at the center ν_0 of the line and $\bar{\kappa}$, the mean of the continuous absorption coefficient over the spectrum. The quantity n is assumed constant through the atmosphere.

We shall consider now the case in which η is not constant but variable through the atmosphere. However, we shall assume the change in η with optical depth to be such that we can write

$$\lambda(t) = \lambda_0 + \delta\lambda(t), \quad (11)$$

where the quantity $\delta\lambda$ represents variations, small compared to λ , about a properly chosen constant value λ_0 . Under the same assumption, the Planck intensity will be given by the expression

$$B(t) = a_{\nu_0} + bt + b_{\nu_0} \int_0^t \delta\lambda dt. \quad (12)$$

We have, accordingly, to solve the system of equations

$$\mu_i \frac{dI_i}{dt} = I_i - \frac{1}{2} (1 - \lambda_0) \sum a_j I_j - \lambda_0 (a_{\nu_0} + bt) + \frac{1}{2} \delta\lambda \sum a_j I_j - \delta\lambda (a_{\nu_0} + bt) - b \int_0^t \delta\lambda dt \quad (i = \pm 1, \dots, \pm n). \quad (13)$$

In solving this system of equations we shall suppose that we can use for the I_j 's, appearing in the second summation term on the right-hand side, the solution $I_j^{(0)}$ obtained in the "first approximation" on the assumption of constant $\lambda = \lambda_0$. We have then to consider the system of nonhomogeneous linear equations

$$\mu_i \frac{dI_i}{dt} = I_i - \frac{1}{2} (1 - \lambda_0) \sum a_j I_j - \lambda_0 (a_{\nu_0} + bt) + \frac{1}{2} \delta\lambda \sum a_j I_j^{(0)} - \delta\lambda (a_{\nu_0} + bt) - b \int_0^t \delta\lambda dt \quad (i = \pm 1, \dots, \pm n). \quad (14)$$

It appears that the most convenient method for solving this system of equations is the method of the variation of the parameters. Thus, writing the general solution in the form (cf. II, eq. [18])

$$I_i(t) = \sum_{a=1}^n \left[\frac{L_a(t) e^{-k_a t}}{1 + \mu_i k_a} + \frac{L_{-a}(t) e^{+k_a t}}{1 - \mu_i k_a} \right] + (\mu_i b + a_{\nu_0} + bt) \quad (i = \pm 1, \dots, \pm n), \quad (15)$$

where L_a and L_{-a} ($a = 1, \dots, n$) are functions of t , we readily obtain the variational equation

$$\mu_i \sum_{a=1}^n \left[\frac{e^{-k_a t}}{1 + \mu_i k_a} \frac{dL_a}{dt} + \frac{e^{+k_a t}}{1 - \mu_i k_a} \frac{dL_{-a}}{dt} \right] = \frac{1}{2} \delta\lambda \sum_{j=-n}^n a_j I_j^{(0)} - \delta\lambda (a_{\nu_0} + bt) - b \int_0^t \delta\lambda dt \quad (i = \pm 1, \dots, \pm n). \quad (16)$$

Substituting for $I_j^{(0)}$ in the foregoing equation from equation (6), we have

$$\mu_i \sum_{a=1}^n \left[\frac{e^{-k_a t}}{1 + \mu_i k_a} \frac{dL_a}{dt} + \frac{e^{+k_a t}}{1 - \mu_i k_a} \frac{dL_{-a}}{dt} \right] = \delta\lambda \left[\frac{1}{1 - \lambda_0} \sum_{a=1}^n L_a^{(0)} e^{-k_a t} \right] - b \int_0^t \delta\lambda dt \quad (i = \pm 1, \dots, \pm n). \quad (17)$$

Equation (17) represents a system of $2n$ linearly independent equations for the $2n$ functions L_a and L_{-a} ($a = 1, \dots, n$).

The system (17) can be solved in the manner presented in a paper by Chandrasekhar⁶ for the solution of a similar set of equations (cf. VII, eq. [46]). Multiplying equation (17) by $a_i \mu_i^{m-1}$ ($m = 1, \dots, 2n$) and summing over all i 's, we obtain

$$\left. \begin{aligned} \sum_{a=1}^n \left[D_{m, a} e^{-k_a t} \frac{dL_a}{dt} + (-1)^m D_{m, a} e^{+k_a t} \frac{dL_{-a}}{dt} \right] \\ = \frac{2}{m} \epsilon_{m, \text{odd}} \Delta \quad (m = 1, \dots, 2n), \end{aligned} \right\} \quad (18)$$

where we have written

$$\Delta = \delta \lambda \left[\frac{1}{1 - \lambda_0} \sum_{a=1}^n L_a^{(0)} e^{-k_a t} \right] - b \int_0^t \delta \lambda dt \quad (19)$$

and

$$D_{m, a} = \sum_i \frac{a_i \mu_i^m}{1 + \mu_i k_a} = (-1)^m \sum_i \frac{a_i \mu_i^m}{1 - \mu_i k_a}. \quad (20)$$

The appearance of the quantity $\epsilon_{m, \text{odd}}$ in equation (18) results from the relation

$$\sum_i a_i \mu_i^{m-1} = \frac{2}{m} \epsilon_{m, \text{odd}} \quad (m = 1, \dots, 4n), \quad (21)$$

where

$$\left. \begin{aligned} \epsilon_{m, \text{odd}} &= 1 \text{ if } m \text{ is odd} \\ &= 0 \text{ otherwise.} \end{aligned} \right\} \quad (22)$$

The quantity $D_{m, a}$ can be evaluated directly by means of the recursion formula

$$D_{m, a} = \frac{1}{k_a} \left(\frac{2}{m} \epsilon_{m, \text{odd}} - D_{m-1, a} \right) \quad (m = 1, \dots, 4n), \quad (23)$$

which for odd, respectively even, values of m takes the forms

$$D_{2j-1, a} = \frac{1}{k_a} \left(\frac{2}{2j-1} - D_{2j-2, a} \right) \quad (24)$$

and

$$D_{2j, a} = -\frac{1}{k_a} D_{2j-1, a}. \quad (25)$$

In this notation the equation for the characteristic roots k_a becomes (cf. eq. [7])

$$D_{0, a} = \frac{2}{1 - \lambda_0}. \quad (26)$$

For odd, respectively even, values of m , equation (18) takes the forms

$$\sum_{a=1}^n D_{2j, a} \left[e^{-k_a t} \frac{dL_a}{dt} + e^{+k_a t} \frac{dL_{-a}}{dt} \right] = 0 \quad (j = 1, \dots, n) \quad (27)$$

and

$$\sum_{a=1}^n D_{2j-1, a} \left[e^{-k_a t} \frac{dL_a}{dt} - e^{+k_a t} \frac{dL_{-a}}{dt} \right] = \frac{2}{2j-1} \Delta \quad (j = 1, \dots, n). \quad (28)$$

⁶ *Ap. J.*, 101, 328, 1945. This paper will be referred to hereafter as "VII."

It is seen that equation (27) represents a system of homogeneous linear equations. Since the determinant of this system clearly does not vanish, we must have

$$e^{-k_a t} \frac{dL_a}{dt} + e^{+k_a t} \frac{dL_{-a}}{dt} = 0 \quad (a = 1, \dots, n). \quad (29)$$

Equation (28) now reduces to

$$\sum_{a=1}^n D_{2j-1, a} e^{-k_a t} \frac{dL_a}{dt} = \frac{1}{2j-1} \Delta \quad (j = 1, \dots, n). \quad (30)$$

Expanding $D_{2j-1, a}$ by means of the recursion formula (23), we have

$$2 \sum_{a=1}^n \left\{ \frac{-D_0}{2k_a^{2j-1}} + \frac{1}{k_a^{2j-1}} + \frac{1}{3k_a^{2j-3}} + \dots + \frac{1}{(2j-1)k_a} \right\} \times X_a = \frac{1}{2j-1} \Delta \quad (j=1, \dots, n), \quad (31)$$

where, for the sake of brevity, we have written

$$X_a = e^{-k_a t} \frac{dL_a}{dt}. \quad (32)$$

We can reduce the system of equations (31) to the simpler one⁷

$$\sum_{a=1}^n \frac{1}{k^{2j-1}} X_a = \frac{1}{2-D_0} U_j \quad (j=1, \dots, n), \quad (33)$$

where the U_j 's are defined as follows:

$$\left. \begin{aligned} U_1 &= \Delta, \\ U_2 &= \frac{1}{3}U_1 - \frac{2}{3(2-D_0)}U_1, \\ U_3 &= \frac{1}{5}U_1 - \frac{2}{3(2-D_0)}U_2 - \frac{2}{5(2-D_0)}U_1, \\ &\dots\dots\dots, \\ U_n &= \frac{1}{2n-1}U_1 - \frac{2}{3(2-D_0)}U_{n-1} - \frac{2}{5(2-D_0)}U_{n-2} - \dots\dots\dots \\ &\quad\quad\quad - \frac{2}{(2n-1)(2-D_0)}U_1. \end{aligned} \right\} \quad (34)$$

The inverse⁸ of the linear transformation which appears on the left-hand side in the system of equations (33) is known. The solution for X_a can, accordingly, be found. We have

$$X_a = \left\{ \begin{array}{l} e^{-k_a t} \frac{dL_a}{dt} \\ - e^{+k_a t} \frac{dL_{-a}}{dt} \end{array} \right\} = \frac{1}{2 - D_0} \frac{k_a^{2n-1}}{\prod_{j \neq a} (k_a^2 - k_j^2)} \sum_{\lambda=0}^{n-1} S_{\lambda, a} U_{\lambda+1} \quad (35)$$

($a = 1, \dots, n$),

⁷ Cf. *ibid.*, p. 337.

⁸ Cf. *ibid.*, pp. 339–341.

where $S_{\lambda,a}$ are the n independent symmetric functions in the $(n-1)$ variables k_μ^2 ($\mu = 1, \dots, r-1, r+1, \dots, n$), defined as in equation (79) of paper VII.

It is convenient to re-write equation (35) in the form

$$e^{-k_a t} \frac{dL_a}{dt} = -e^{+k_a t} \frac{dL_{-a}}{dt} = (1 - \lambda_0) Q_a \Delta \quad (\alpha = 1, \dots, n), \quad (36)$$

where the Q_a 's are defined by

$$Q_a = -\frac{1}{2\lambda_0 \Delta} \frac{k_a^{2n-1}}{\prod_{j \neq a} (k_a^2 - k_j^2)} \sum_{\lambda=0}^{n-1} S_{\lambda,a} U_{\lambda+1} \quad (\alpha = 1, \dots, n). \quad (37)$$

Substituting for Δ from equation (19) in equation (36), we obtain

$$\frac{dL_a}{dt} = Q_a \left[\delta \lambda \sum_{\beta=1}^n L_{\beta}^{(0)} e^{-k_{\beta} t} - (1 - \lambda_0) b \int_0^t \delta \lambda dt \right] e^{+k_a t} \quad (38)$$

and

$$\frac{dL_{-a}}{dt} = -Q_a \left[\delta \lambda \sum_{\beta=1}^n L_{\beta}^{(0)} e^{-k_{\beta} t} - (1 - \lambda_0) b \int_0^t \delta \lambda dt \right] e^{-k_a t}, \quad (39)$$

from which the solutions for L_a and L_{-a} readily follow. We have

$$\left. \begin{aligned} L_a(t) = Q_a \left[\int_0^t \delta \lambda \left(\sum_{\beta=1}^n L_{\beta}^{(0)} e^{-k_{\beta} t} \right) e^{+k_a t} dt - \right. \\ \left. (1 - \lambda_0) b \int_0^t e^{+k_a t} \left(\int_0^t \delta \lambda dt \right) dt \right] + L_a^{(0)} - \gamma_a \quad (\alpha = 1, \dots, n), \end{aligned} \right\} \quad (40)$$

where γ_a ($\alpha = 1, \dots, n$) are n constants of integration; and

$$\left. \begin{aligned} L_{-a}(t) = Q_a \left[\int_t^\infty \delta \lambda \left(\sum_{\beta=1}^n L_{\beta}^{(0)} e^{-k_{\beta} t} \right) e^{-k_a t} dt - (1 - \lambda_0) b \int_t^\infty e^{-k_a t} \right. \\ \left. \times \left(\int_0^t \delta \lambda dt \right) dt \right] \quad (\alpha = 1, \dots, n). \end{aligned} \right\} \quad (41)$$

It will be noticed that in integrating equation (39) in the form (41) the constants of integration have been so chosen as to satisfy the boundary condition that none of the quantities increase exponentially as $\tau \rightarrow \infty$. At $t = 0$ the equations (40) and (41) become

$$L_a(0) = L_a^{(0)} - \gamma_a \quad (\alpha = 1, \dots, n) \quad (42)$$

and

$$\left. \begin{aligned} L_{-a}(0) = Q_a \left[\sum_{\beta=1}^n \left(\frac{L_{\beta}^{(0)}}{k_a + k_{\beta}} \int_0^\infty \delta \lambda e^{-(k_a + k_{\beta}) t} dt (k_a + k_{\beta}) t \right) \right. \\ \left. - \frac{(1 - \lambda_0) b}{k_a^2} \int_0^\infty \delta \lambda e^{-k_a t} dt (k_a t) \right] \quad (\alpha = 1, \dots, n). \end{aligned} \right\} \quad (43)$$

The latter equation can be written in the form

$$L_{-a}(0) = Q_a \left[\sum_{\beta=1}^n \frac{L_{\beta}^{(0)}}{k_a + k_{\beta}} \frac{\overline{\delta \lambda}_{k_a + k_{\beta}}}{\overline{\delta \lambda}_{k_a}} - \frac{(1 - \lambda_0) b}{k_a^2} \frac{\overline{\delta \lambda}_{k_a}}{\overline{\delta \lambda}_{k_a}} \right] \quad (a = 1, \dots, n), \quad (44)$$

where we have introduced the quantities

$$\overline{\delta \lambda}_{k_a + k_{\beta}} = \int_0^{\infty} \delta \lambda e^{-(k_a + k_{\beta})t} d(k_a + k_{\beta}) t \quad (45)$$

and

$$\overline{\delta \lambda}_{k_a} = \int_0^{\infty} \delta \lambda e^{-k_a t} d(k_a t). \quad (46)$$

Equations (45) and (46) represent certain averages over the values of $\delta \lambda$ through the atmosphere, weighted according to the functions $\exp[-(k_a + k_{\beta})t]$ and $\exp[-k_a t]$, respectively.

The constants of integration γ_a ($a = 1, \dots, n$) which occur in the solution for $L_a(0)$ (cf. eq. [42]) are determined from the further boundary condition that there is no incident radiation on the surface $t = 0$:

$$I_{-i} = 0 \text{ at } t = 0 \text{ for } i = 1, \dots, n. \quad (47)$$

This condition requires that (cf. eq. [15])

$$\sum_{a=1}^n \left[\frac{L_a^{(0)} - \gamma_a}{1 - \mu_i k_a} + \frac{L_{-a}(0)}{1 + \mu_i k_a} \right] + a_{v_0} = \mu_i b \quad (i = 1, \dots, n), \quad (48)$$

or, since $L_a^{(0)}$ satisfies the relation (cf. eq. [8])

$$\sum_{a=1}^n \frac{L_a^{(0)}}{1 - \mu_i k_a} + a_{v_0} = \mu_i b \quad (i = 1, \dots, n), \quad (49)$$

the equations for γ_a are

$$\sum_{a=1}^n \frac{\gamma_a}{1 - \mu_i k_a} = \sum_{a=1}^n \frac{L_{-a}(0)}{1 + \mu_i k_a} \quad (i = 1, \dots, n). \quad (50)$$

In terms of the matrices

$$\mathbf{G} = (G_{ij}) = \frac{1}{1 - \mu_i k_j} \quad \text{and} \quad \mathbf{H} = (H_{ij}) = \frac{1}{1 + \mu_i k_j} \quad (51)$$

and the vectors

$$\mathbf{\gamma} = (\gamma_a) \quad \text{and} \quad \mathbf{L}_{-} = (L_{-a}), \quad (52)$$

the formal solution to the foregoing system of equations is

$$\mathbf{\gamma} = \mathbf{G}^{-1} \mathbf{H} \mathbf{L}_{-}, \quad (53)$$

where the inverse of the matrix $\mathbf{G}$ is known explicitly.⁹

This completes the solution to the variational equations (17) in the n th approximation.

The expression for the emergent intensity becomes

$$I_i(0) = \sum_{a=1}^n \left[\frac{L_a^{(0)} - \gamma_a}{1 + \mu_i k_a} + \frac{L_{-a}(0)}{1 - \mu_i k_a} \right] + \mu_i b + a_{v_0} \quad (i = 1, \dots, n). \quad (54)$$

⁹ Cf. VI, pp. 320-322.

The foregoing equation can be written alternatively in the form

$$I_i(0) = \sum_{a=1}^n \left[c_{i,a} \frac{L_a^{(0)}}{1-\lambda_0} + d_{i,a} \frac{L_{-a}(0)}{1-\lambda_0} \right] + \mu_i b + a_{\nu_0} \quad (i = 1, \dots, n), \quad (55)$$

where

$$c_{i,a} = \frac{1-\lambda_0}{1+\mu_i k_a} \quad \text{and} \quad d_{i,a} = \frac{1-\lambda_0}{1-\mu_i k_a} \left[1 - \frac{1-\mu_i k_a}{1+\mu_i k_a} \frac{\gamma_a}{L_{-a}(0)} \right]. \quad (56)$$

The emergent intensity in the continuous spectrum is given by

$$I_i(0, \text{cont.}) = a_{\nu_0} + b_{\nu_0} \mu_i \quad (i = 1, \dots, n), \quad (57)$$

obtained by putting $\lambda = \lambda_0 = 1$, in equation (55). Finally, the residual intensity in the line

$$r_i = \frac{I_i(0)}{I_i(0, \text{cont.})} \quad (i = 1, \dots, n) \quad (58)$$

can be written in the form

$$\left(\frac{8}{3} x + \mu_i \right) r_i = \sum_{a=1}^n [c_{i,a} \mathfrak{L}'_a + d_{i,a} \mathfrak{L}'_{-a}(0)] + \lambda_0 \mu_i + \frac{8}{3} x \quad (i = 1, \dots, n), \quad (59)$$

where

$$\mathfrak{L}'_a = \frac{\lambda_0 L_a^{(0)}}{b(1-\lambda_0)} \quad (60)$$

and

$$\mathfrak{L}'_{-a}(0) = Q_a \left[\sum_{\beta=1}^n \frac{\mathfrak{L}'_{\beta}}{k_a + k_{\beta}} \frac{\overline{\delta \lambda}_{k_a + k_{\beta}}}{k_{\beta}^2} - \frac{\lambda_0}{k_a^2} \frac{\overline{\delta \lambda}_{k_a}}{\delta \lambda_{k_a}} \right]. \quad (61)$$

Equation (59) enables us, then, to compute in the n th approximation theoretical contours for n different points along a radius of the stellar disc, the emergent radiation being assumed to be divided into n streams symmetrically located along each radius.

It may be recalled that in the integrals over $\delta \lambda$ we can put

$$t_{\nu} = \frac{n_{\nu_0}}{\lambda_{0_{\nu}}} \tau. \quad (62)$$

We have

$$\overline{\delta \lambda}_{k_a + k_{\beta}} = \int_0^{\infty} \delta \lambda e^{-(k_a + k_{\beta}) n_{\nu_0} \tau / \lambda_0} d \left[(k_a + k_{\beta}) \frac{n_{\nu_0}}{\lambda_0} \tau \right] \quad (63)$$

and

$$\overline{\delta \lambda}_{k_a} = \int_0^{\infty} \delta \lambda e^{-k_a n_{\nu_0} \tau / \lambda_0} d \left[k_a \frac{n_{\nu_0}}{\lambda_0} \tau \right]. \quad (64)$$

The difference in the weighting functions for $\overline{\delta \lambda}_{k_a}$ and $\overline{\delta \lambda}_{k_a + k_{\beta}}$ is related to the difference between the two mechanisms operative in the formation of absorption lines. Absorption lines are produced by continuous absorption, depending upon the existence of a temperature gradient, and by scattering, causing a deviation from Kirchhoff's law. The integral $\overline{\delta \lambda}_{k_a}$, to which relatively deeper layers contribute, is associated with the first of these effects; the integral $\overline{\delta \lambda}_{k_a + k_{\beta}}$, to which relatively higher layers contribute, is associated with the second.

3. *The solutions in the first and second approximations: (a) The first approximation.*—The discussion of the approximation $n = 1$ is of particular interest, since the solution obtained can be compared directly with Strömgren's¹⁰ work. For this case we have $\mu_1 = 1/\sqrt{3}$ and $k_1 = \sqrt{(3\lambda_0)}$. The final expression for the emergent intensity becomes

$$I_1(0) = \left. \begin{aligned} & \frac{2a_{v_0}\sqrt{\lambda_0} + \frac{2}{\sqrt{3}}b}{1 + \sqrt{\lambda_0}} + \frac{\frac{a_{v_0}}{\sqrt{\lambda_0}} - \frac{b}{\sqrt{3\lambda_0}}}{(1 + \sqrt{\lambda_0})^2} \int_0^\infty \delta\lambda e^{-2\sqrt{3\lambda_0}t} d(2\sqrt{3\lambda_0}t) \\ & + \frac{2b}{1 + \sqrt{\lambda_0}} \int_0^\infty \delta\lambda e^{-\sqrt{3\lambda_0}t} d(\sqrt{3\lambda_0}t) \end{aligned} \right\} \quad (65)$$

Performing the transformations carried out by Strömgren,¹¹ we can reduce equation (65) to the form

$$I_1(0) = \frac{2a_{v_0}\sqrt{\bar{\lambda}} + \frac{2}{\sqrt{3}}b_{v_0}\bar{\lambda}}{1 + \sqrt{\bar{\lambda}}}, \quad (66)$$

where

$$\bar{\lambda} = \int_0^\infty \lambda e^{-\sqrt{3\lambda_0}t} d(\sqrt{3\lambda_0}t) \quad (67)$$

and

$$\sqrt{\bar{\lambda}} = \int_0^\infty \sqrt{\lambda} e^{-2\sqrt{3\lambda_0}t} d(2\sqrt{3\lambda_0}t). \quad (68)$$

Using the relation (cf. IV, eq. [7])

$$F(t=0) = \frac{2}{\sqrt{3}} I_1(t=0), \quad (69)$$

we have

$$F(t=0) = \frac{\frac{4}{3}a_{v_0}\sqrt{3}\sqrt{\bar{\lambda}} + \frac{4}{3}b_{v_0}\bar{\lambda}}{1 + \sqrt{\bar{\lambda}}}. \quad (70)$$

This is equivalent to Strömgren's result (cf. *op. cit.*, eq. [87]) except for the omission of the factor $2/\sqrt{3}$ before the term $\sqrt{\bar{\lambda}}$ in the denominator. This difference results from the fact that the Eddington approximation, $J = \frac{1}{2}F$ at the surface, is not made in the present work.

b) The second approximation.—In this approximation $\mu_1 = 0.34$ and $\mu_2 = 0.86$, and theoretical contours can accordingly be computed for these points. The final expressions for the emergent intensity are

$$I_1(0) = \left. \begin{aligned} & \frac{L_1^{(0)}}{1 + \mu_1 k_1} + \frac{L_2^{(0)}}{1 + \mu_1 k_2} + \mu_1 b + a_{v_0} \\ & + \frac{4\mu_1 k_1 (k_1 + k_2) (\mu_1 + \mu_2)}{(1 + \mu_1 k_1)^2 (1 + \mu_2 k_1) (1 + \mu_1 k_2) (1 - \mu_1 k_1)} L_{-1}(0) \\ & + \frac{4\mu_1 k_2 (k_1 + k_2) (\mu_1 + \mu_2)}{(1 + \mu_1 k_2)^2 (1 + \mu_2 k_2) (1 + \mu_1 k_1) (1 - \mu_1 k_2)} L_{-2}(0) \end{aligned} \right\} \quad (71)$$

¹⁰ *Op. cit.* (see n. 1, above).

¹¹ *Ibid.*, pp. 14–16.

and

$$I_2(0) = \frac{L_1^{(0)}}{1 + \mu_2 \bar{k}_1} + \frac{L_2^{(0)}}{1 + \mu_2 \bar{k}_2} + \mu_2 b + a_{v_0} \left. \begin{aligned} &+ \frac{4\mu_2 \bar{k}_1 (k_1 + k_2) (\mu_1 + \mu_2)}{(1 + \mu_2 \bar{k}_1)^2 (1 + \mu_1 \bar{k}_1) (1 + \mu_2 \bar{k}_2) (1 - \mu_2 \bar{k}_1)} L_{-1}(0) \\ &+ \frac{4\mu_2 \bar{k}_2 (k_1 + k_2) (\mu_1 + \mu_2)}{(1 + \mu_2 \bar{k}_2)^2 (1 + \mu_1 \bar{k}_2) (1 + \mu_2 \bar{k}_1) (1 - \mu_2 \bar{k}_2)} L_{-2}(0) \end{aligned} \right\} \quad (72)$$

The quantities $L_1^{(0)}$ and $L_2^{(0)}$ are given by (cf. VI, eq. [11])

$$L_1^{(0)} = \frac{(1 - \mu_1 \bar{k}_1) (1 - \mu_2 \bar{k}_1)}{\bar{k}_1 - \bar{k}_2} b \left[1 - k_2 (\mu_1 + \mu_2) + \frac{a_{v_0}}{b} k_2 \right] \quad (73)$$

and

$$L_2^{(0)} = \frac{(1 - \mu_2 \bar{k}_2) (1 - \mu_1 \bar{k}_2)}{\bar{k}_2 - \bar{k}_1} b \left[1 - k_1 (\mu_1 + \mu_2) + \frac{a_{v_0}}{b} k_1 \right]. \quad (74)$$

The solutions for $L_{-1}(0)$ and $L_{-2}(0)$ are

$$L_{-1}(0) = \frac{-\bar{k}_1^3}{2\lambda_0 (\bar{k}_1^2 - \bar{k}_2^2)} \left(1 - \frac{\bar{k}_2^2}{3\lambda_0} \right) \times \left[\frac{L_1^{(0)}}{2\bar{k}_1} \overline{\delta\lambda_{2k_1}} + \frac{L_2^{(0)}}{\bar{k}_1 + \bar{k}_2} \overline{\delta\lambda_{k_1+k_2}} - (1 - \lambda_0) \frac{b}{\bar{k}_1^2} \overline{\delta\lambda_{k_1}} \right] \quad (75)$$

and

$$L_{-2}(0) = \frac{-\bar{k}_2^3}{2\lambda_0 (\bar{k}_2^2 - \bar{k}_1^2)} \left(1 - \frac{\bar{k}_1^2}{3\lambda_0} \right) \times \left[\frac{L_1^{(0)}}{\bar{k}_1 + \bar{k}_2} \overline{\delta\lambda_{k_1+k_2}} + \frac{L_2^{(0)}}{2\bar{k}_2} \overline{\delta\lambda_{2k_2}} - (1 - \lambda_0) \frac{b}{\bar{k}_2^2} \overline{\delta\lambda_{k_2}} \right]. \quad (76)$$

4. *The solution in the third approximation in numerical form.*—In this approximation the points of the Gaussian division are $\mu_1 = 0.24$, $\mu_2 = 0.66$, and $\mu_3 = 0.93$. Accordingly, contours for these points, representing the fractional parts of the radius, 0.97, 0.75, and 0.36, respectively, from the center of the disc can be obtained. The expressions for the emergent intensities at these points become

$$I_1(0) = \frac{L_1^{(0)}}{1 + \mu_1 \bar{k}_1} + \frac{L_2^{(0)}}{1 + \mu_1 \bar{k}_2} + \frac{L_3^{(0)}}{1 + \mu_1 \bar{k}_3} + \mu_1 b + a_{v_0} \left. \begin{aligned} &+ \frac{4\mu_1 \bar{k}_1 (k_1 + k_2) (k_3 + k_1) (\mu_1 + \mu_2) (\mu_3 + \mu_1)}{(1 + \mu_1 \bar{k}_1)^2 (1 + \mu_2 \bar{k}_1) (1 + \mu_3 \bar{k}_1) (1 + \mu_1 \bar{k}_2) (1 + \mu_1 \bar{k}_3) (1 - \mu_1 \bar{k}_1)} L_{-1}(0) \\ &+ \frac{4\mu_1 \bar{k}_2 (k_1 + k_2) (k_2 + k_3) (\mu_1 + \mu_2) (\mu_3 + \mu_1)}{(1 + \mu_1 \bar{k}_2)^2 (1 + \mu_2 \bar{k}_2) (1 + \mu_3 \bar{k}_2) (1 + \mu_1 \bar{k}_1) (1 + \mu_1 \bar{k}_3) (1 - \mu_1 \bar{k}_2)} L_{-2}(0) \\ &+ \frac{4\mu_1 \bar{k}_3 (k_3 + k_1) (k_2 + k_3) (\mu_1 + \mu_2) (\mu_3 + \mu_1)}{(1 + \mu_1 \bar{k}_3)^2 (1 + \mu_2 \bar{k}_3) (1 + \mu_3 \bar{k}_3) (1 + \mu_1 \bar{k}_1) (1 + \mu_1 \bar{k}_2) (1 - \mu_1 \bar{k}_3)} L_{-3}(0) \end{aligned} \right\} \quad (77)$$

$$\begin{aligned}
 I_2(0) = & \frac{L_1^{(0)}}{1+\mu_2 k_1} + \frac{L_2^{(0)}}{1+\mu_2 k_2} + \frac{L_3^{(0)}}{1+\mu_2 k_3} + \mu_2 b + a_{\nu_0} \\
 & + \frac{4\mu_2 k_1 (k_1+k_2) (k_3+k_1) (\mu_1+\mu_2) (\mu_2+\mu_3)}{(1+\mu_1 k_1) (1+\mu_2 k_1)^2 (1+\mu_3 k_1) (1+\mu_2 k_2) (1+\mu_2 k_3) (1-\mu_2 k_1)} L_{-1}(0) \\
 & + \frac{4\mu_2 k_2 (k_1+k_2) (k_2+k_3) (\mu_1+\mu_2) (\mu_2+\mu_3)}{(1+\mu_1 k_2) (1+\mu_2 k_2)^2 (1+\mu_3 k_2) (1+\mu_2 k_1) (1+\mu_2 k_3) (1-\mu_2 k_2)} L_{-2}(0) \\
 & + \frac{4\mu_2 k_3 (k_3+k_1) (k_2+k_3) (\mu_1+\mu_2) (\mu_2+\mu_3)}{(1+\mu_1 k_3) (1+\mu_2 k_3)^2 (1+\mu_3 k_3) (1+\mu_2 k_1) (1+\mu_2 k_2) (1-\mu_2 k_3)} L_{-3}(0),
 \end{aligned} \quad (78)$$

and

$$\begin{aligned}
 I_3(0) = & \frac{L_1^{(0)}}{1+\mu_3 k_1} + \frac{L_2^{(0)}}{1+\mu_3 k_2} + \frac{L_3^{(0)}}{1+\mu_3 k_3} + \mu_3 b + a_{\nu_0} \\
 & + \frac{4\mu_3 k_1 (k_1+k_2) (k_3+k_1) (\mu_3+\mu_1) (\mu_2+\mu_3)}{(1+\mu_1 k_1) (1+\mu_2 k_1) (1+\mu_3 k_1)^2 (1+\mu_3 k_2) (1+\mu_3 k_3) (1-\mu_3 k_1)} L_{-1}(0) \\
 & + \frac{4\mu_3 k_2 (k_1+k_2) (k_2+k_3) (\mu_3+\mu_1) (\mu_2+\mu_3)}{(1+\mu_1 k_2) (1+\mu_2 k_2) (1+\mu_3 k_2)^2 (1+\mu_3 k_1) (1+\mu_3 k_3) (1-\mu_3 k_2)} L_{-2}(0) \\
 & + \frac{4\mu_3 k_3 (k_3+k_1) (k_2+k_3) (\mu_3+\mu_1) (\mu_2+\mu_3)}{(1+\mu_1 k_3) (1+\mu_2 k_3) (1+\mu_3 k_3)^2 (1+\mu_3 k_1) (1+\mu_3 k_2) (1-\mu_3 k_3)} L_{-3}(0).
 \end{aligned} \quad (79)$$

The equations for the constants $L_a^{(0)}$ are (cf. VI, eq. [9])

$$L_1^{(0)} = \frac{(1-\mu_1 k_1) (1-\mu_2 k_1) (1-\mu_3 k_1)}{(k_1-k_2) (k_3-k_1)} b \left[(k_2+k_3) - k_2 k_3 (\mu_1+\mu_2+\mu_3) + \frac{a_{\nu_0}}{b} k_2 k_3 \right], \quad (80)$$

$$L_2^{(0)} = \frac{(1-\mu_1 k_2) (1-\mu_2 k_2) (1-\mu_3 k_2)}{(k_2-k_3) (k_1-k_2)} b \left[(k_3+k_1) - k_3 k_1 (\mu_1+\mu_2+\mu_3) + \frac{a_{\nu_0}}{b} k_3 k_1 \right], \quad (81)$$

and

$$L_3^{(0)} = \frac{(1-\mu_1 k_3) (1-\mu_2 k_3) (1-\mu_3 k_3)}{(k_3-k_1) (k_2-k_3)} b \left[(k_1+k_2) - k_1 k_2 (\mu_1+\mu_2+\mu_3) + \frac{a_{\nu_0}}{b} k_1 k_2 \right]. \quad (82)$$

The quantities $L_{-a}(0)$ are given by

$$L_{-1}(0) = Q_1 \left[\frac{L_1^{(0)}}{2k_1} \overline{\delta \lambda_{2k_1}} + \frac{L_2^{(0)}}{k_1+k_2} \overline{\delta \lambda_{k_1+k_2}} + \frac{L_3^{(0)}}{k_1+k_3} \overline{\delta \lambda_{k_1+k_3}} - (1-\lambda_0) \frac{b}{k_1^2} \overline{\delta \lambda_{k_1}} \right], \quad (83)$$

$$L_{-2}(0) = Q_2 \left[\frac{L_1^{(0)}}{k_1+k_2} \overline{\delta \lambda_{k_1+k_2}} + \frac{L_2^{(0)}}{2k_2} \overline{\delta \lambda_{2k_2}} + \frac{L_3^{(0)}}{k_2+k_3} \overline{\delta \lambda_{k_2+k_3}} - (1-\lambda_0) \frac{b}{k_2^2} \overline{\delta \lambda_{k_2}} \right], \quad (84)$$

and

$$L_{-3}(0) = Q_3 \left[\frac{L_1^{(0)}}{k_1+k_3} \overline{\delta \lambda_{k_1+k_3}} + \frac{L_2^{(0)}}{k_2+k_3} \overline{\delta \lambda_{k_2+k_3}} + \frac{L_3^{(0)}}{2k_3} \overline{\delta \lambda_{2k_3}} - (1-\lambda_0) \frac{b}{k_3^2} \overline{\delta \lambda_{k_3}} \right], \quad (85)$$

where

$$Q_1 = \frac{k_1^5}{2\lambda_0 (k_1^2 - k_2^2) (k_3^2 - k_1^2)} \left[1 - \frac{k_2^2 + k_3^2}{3\lambda_0} + \left(\frac{1}{5\lambda_0} + \frac{1 - \lambda_0}{9\lambda_0^2} \right) k_2^2 k_3^2 \right], \quad (86)$$

$$Q_2 = \frac{k_2^5}{2\lambda_0 (k_1^2 - k_2^2) (k_2^2 - k_3^2)} \left[1 - \frac{k_1^2 + k_3^2}{3\lambda_0} + \left(\frac{1}{5\lambda_0} + \frac{1 - \lambda_0}{9\lambda_0^2} \right) k_1^2 k_3^2 \right], \quad (87)$$

and

$$Q_3 = \frac{k_3^5}{2\lambda_0 (k_3^2 - k_1^2) (k_2^2 - k_3^2)} \left[1 - \frac{k_1^2 + k_2^2}{3\lambda_0} + \left(\frac{1}{5\lambda_0} + \frac{1 - \lambda_0}{9\lambda_0^2} \right) k_1^2 k_2^2 \right]. \quad (88)$$

The expressions for the residual intensities have the form

$$\left. \begin{aligned} (\frac{8}{3}x + \mu_i) r_i = c_{i,1} \mathfrak{R}'_1 + c_{i,2} \mathfrak{R}'_2 + c_{i,3} \mathfrak{R}'_3 + d_{i,1} \mathfrak{R}'_{-1}(0) \\ + d_{i,2} \mathfrak{R}'_{-2}(0) + d_{i,3} \mathfrak{R}'_{-3}(0) + \lambda_0 \mu_i + \frac{8}{3}x \quad (i = 1, 2, 3) \end{aligned} \right\} \quad (89)$$

where $\mathfrak{R}'_a = \mathfrak{R}_a / (1 - \lambda_0)$. It is the quantities $\mathfrak{R}_a$ which are tabulated for different values of λ_0 and x in paper VI (Table 1). The coefficients $c_{i,a}$ and $d_{i,a}$ are equivalent to the corresponding coefficients of $L_a^{(0)}$ and $L_{-a}(0)$, respectively, in the expressions (77), (78),

TABLE 1
VALUES OF $c_{1,a}$ AND $d_{1,a}$

λ_0	$c_{1,1}$	$c_{1,2}$	$c_{1,3}$	$d_{1,1}$	$d_{1,2}$	$d_{1,3}$
0.....	0.5668	1	0.7738	3.877	0	0.8268
0.10.....	.5037	0.7997	.6944	4.022	0.6235	.9639
0.20.....	.4421	0.6840	.6150	4.071	.7078	.9320
0.30.....	.3818	0.5844	.5356	4.105	.7049	.8652
0.40.....	.3231	0.4930	.4566	4.135	.6564	.7782
0.50.....	.2658	0.4064	.3782	4.160	.5800	.6760
0.60.....	.2100	0.3228	.3006	4.184	.4848	.5608
0.70.....	.1555	0.2409	.2241	4.208	.3759	.4344
0.80.....	.1024	0.1600	.1485	4.230	.2576	.2980
0.90.....	0.0506	0.0798	0.0738	4.253	0.1318	0.1529
1.00.....	0	0	0	4.274	0	0

and (79) for the emergent intensities, multiplied by the factor $(1 - \lambda_0)$. Values of these coefficients $c_{i,a}$ and $d_{i,a}$ for various values of λ_0 are tabulated in Tables 1, 2, and 3. The quantities Q_a have also been computed for different values of λ_0 , the results being given in Table 4. This completes the formal solution to our problem. We now turn to applications of this theory.

5. *The variation of η through the solar atmosphere.*—For each selected point on the contour of the absorption line, the basis of the calculation is the function $\eta(\tau)$, which can be evaluated numerically in the manner of Strömgren.¹² For transitions arising from a ground state, the line absorption coefficient which results as a consequence of radiation damping, collision broadening, and Doppler broadening is given by

$$\sigma = k_0 N_1 H(a, v), \quad (90)$$

¹² *Festschrift für E. Strömgren*, Kopenhagen: Einar Munksgaard, 1940.

TABLE 2
VALUES OF $c_{2,\alpha}$ AND $d_{2,\alpha}$

λ_0	$c_{2,1}$	$c_{2,2}$	$c_{2,3}$	$d_{2,1}$	$d_{2,2}$	$d_{2,3}$
0.....	0.3207	1	0.5524	-1.2454	0	4.694
0.10.....	.2830	0.6680	.4944	0.9162	1.0530	4.764
0.20.....	.2466	0.5443	.4362	0.7400	1.3168	4.654
0.30.....	.2115	0.4521	.3783	0.5977	1.4287	4.575
0.40.....	.1778	0.3746	.3208	0.4768	1.4292	4.556
0.50.....	.1453	0.3053	.2642	0.3720	1.3335	4.604
0.60.....	.1140	0.2405	.2088	0.2798	1.1596	4.715
0.70.....	.0839	0.1785	.1547	0.1980	0.9249	4.876
0.80.....	.0549	0.1181	.1020	0.1250	0.6456	5.074
0.90.....	0.0270	0.0588	0.0505	-0.0593	0.3344	5.298
1.00.....	0	0	0	0	0	5.544

TABLE 3
VALUES OF $c_{3,\alpha}$ AND $d_{3,\alpha}$

λ_0	$c_{3,1}$	$c_{3,2}$	$c_{3,3}$	$d_{3,1}$	$d_{3,2}$	$d_{3,3}$
0.....	0.2508	1	0.4667	-0.8498	0	-7.5841
0.10.....	.2210	0.6041	.4172	.5699	1.473	5.9517
0.20.....	.1922	0.4812	.3677	.4530	2.203	4.6880
0.30.....	.1645	0.3948	.3183	.3635	2.982	3.6022
0.40.....	.1380	0.3247	.2694	.2892	3.887	2.6970
0.50.....	.1126	0.2632	.2214	.2254	4.938	1.9690
0.60.....	.0882	0.2067	.1746	.1696	6.121	1.3936
0.70.....	.0648	0.1531	.1291	.1201	7.414	0.9372
0.80.....	.0423	0.1012	.0849	.0759	8.783	0.5680
0.90.....	0.0208	0.0502	0.0420	-0.0361	10.214	-0.2617
1.00.....	0	0	0	0	11.674	0

TABLE 4
VALUES OF Q_α

λ_0	Q_1	Q_2	Q_3
0.10.....	-0.9519	-2.4004	-0.1773
0.20.....	0.9750	1.4449	.2254
0.30.....	0.9914	0.9679	.2769
0.40.....	1.0014	0.6617	.3206
0.50.....	1.0061	0.4539	.3468
0.60.....	1.0065	0.3142	.3525
0.70.....	1.0034	0.2219	.3420
0.80.....	0.9977	0.1608	.3221
0.90.....	0.9898	0.1199	.2979
1.00.....	-0.9805	-0.0919	-0.2728

where k_0 , the fictitious atomic absorption coefficient at the center of the line for vanishing damping, depends on the Einstein coefficient A_{k1} , corresponding to the transition in question, and where $H(a, v)$ is a certain definite integral involving a and v as two parameters.¹³ The number of absorbing atoms N_1 per gram at a given optical depth τ can be determined with the aid of Saha's equation in terms of the temperature T and the electron pressure p_e prevailing at τ and certain relative abundances, including the ratio of hydrogen atoms to metal atoms (A in Strömgren's notation) and the ratio of the particular absorbing atoms to the metal atoms. The integral defining $H(a, v)$ has been tabulated by Hjerting¹³ for various values of v , representing the deviation from the center of the line in units of the Doppler width, and

$$a = \frac{\Delta\nu_N + \Delta\nu_c}{\Delta\nu_D}. \quad (91)$$

In equation (91) $\Delta\nu_N$ is the coefficient of radiation damping, $\Delta\nu_c$, the coefficient of collision broadening, and $\Delta\nu_D$, the Doppler width. The transition probability A_{k1} determines $\Delta\nu_N$. In evaluating $\Delta\nu_c$ in the solar atmosphere, we shall follow Strömgren and consider only the effect of collisions with neutral hydrogen. Moreover, we shall put the partial pressure of the neutral hydrogen atoms equal to the total pressure P . Under these conditions the number of broadening collisions is proportional to $T^{-0.7}P$ and is related, in addition, to $\overline{R_k^2}$, the mean square radius corresponding to the upper stationary state of

Thus, the evaluation of

$$\eta = \frac{\sigma}{\kappa} = \frac{\sigma}{n\kappa} \quad (92)$$

requires the adoption of a model solar atmosphere, giving the temperature, the total pressure, and the electron pressure as functions of τ , computed with a certain mean absorption coefficient for various assumed values of the hydrogen-metal ratio and the relative abundance of the absorbing atoms. The present investigation will be limited to those lines for which reasonably accurate transition probabilities and the physical data for the calculation of the collision broadening are known.

From the numerical values of $\lambda = 1/1 + \eta$ so derived, appropriate values of λ_0 , about which the relative variations of λ in the relevant range of τ are small, are chosen. It should be mentioned here that the final results are not particularly dependent on the choice of the precise value of λ_0 . For two values of λ_0 which do not differ very greatly lead to the same results. This agreement is, in fact, guaranteed by the circumstance that our theory is correct to quantities of the first order. The proper averages of $\delta\lambda$ (cf. eqs. [63] and [64]) can then be evaluated by numerical quadratures. In carrying out these averages it was found convenient and sufficiently accurate to use a three-point integration formula,¹⁴ according to which we set the integral

$$\overline{\delta\lambda_{f(k)}} = \int_0^\infty \delta\lambda e^{-f(k)n\tau/\lambda_0} d\left[f(k) \frac{n}{\lambda_0} \tau\right] = \sum_{i=1}^3 q_i \delta\lambda(x_i), \quad (93)$$

where $x_i = [f(k)n\tau/\lambda_0]$ ($i = 1, 2, 3$) are the roots of the Laguerre polynomial $L_3(x)$:

$$x_1 = 0.41577, \quad x_2 = 2.2943, \quad x_3 = 6.2899. \quad (94)$$

¹³ F. Hjerting, *Ap. J.*, **88**, 508, 1938.

¹⁴ A. Reiz, *Arkiv för matematik, astronomi och fysik*, Vol. **29**, Part 4, 1944.

The corresponding "weights" q_i are

$$q_1 = 0.71109, \quad q_2 = 0.27852, \quad q_3 = 0.010389. \quad (95)$$

Thus, $\overline{\delta\lambda}_{f(k)}$ is determined as the weighted mean of the values of $\delta\lambda$ at

$$\tau_i = \frac{1}{f(k)} \frac{\lambda_0}{n} x_i \quad (i = 1, 2, 3). \quad (96)$$

6. *The comparison of theoretical contours with observed contours.*—In the present investigation a model solar atmosphere in radiative equilibrium, computed by Rudkjøbing¹⁵ for a boundary temperature $\theta_0 = 5040/T_0 = 1.041$ and $\log g = 4.44$ and for the hydrogen-metal ratio $\log A = 3.8$, obtained by Strömberg,¹² was used. The mean absorption coefficient $\bar{\kappa}$ corresponds to the Rosseland mean over all frequencies of the sum of the continuous absorption of neutral hydrogen and the negative hydrogen ion, set equal to its maximum value $2.6 \times 10^{-17} \text{ cm}^2$, as derived by Massey and Bates,¹⁶ over the entire spectrum.

Houtgast's¹⁷ observations of strong Fraunhofer lines at seven and eight points along a solar radius provide the most complete basis for the comparison of the theory with observations. Of the lines observed by Houtgast, those selected for application of the present theory of stellar absorption lines include the D lines of sodium, the H and K lines of ionized calcium, and the calcium line $\lambda 4227$. Strömberg's¹² relative abundances of sodium and calcium in the sun were adopted; the logarithm of the number of sodium atoms per gram matter is 17.7; of calcium atoms, 18.0, the manner of determination of the latter being more approximate. Values of n , to which the calculation of the residual intensities is sensitive, were obtained from Münch's¹⁸ discussion of the observed emergent intensity distribution at the center of the solar disc and the emergent flux distribution of the continuous spectrum of the sun. For the H and K lines of Ca II, n is 0.674 and 0.694, respectively; for the Ca I line $\lambda 4227$, $n = 0.636$; and for the D₁ and D₂ lines of Na I, $n = 0.756$ and 0.755, respectively. For the sodium D lines the transition probability was adopted according to Ladenburg and Thiele,¹⁹ $A_{k1} = 6.8 \times 10^7$, and the mean square radius, needed for the computation of the collision broadening, was derived from wave functions obtained by Fock and Petrashen,²⁰ $\overline{R}_k^2 = 41a_0^2$, a_0 being the atomic unit of length. For the calcium lines, the transition probabilities obtained by Hartree and Hartree²¹ were used; further, from their wave functions the mean square radii were determined. We have for the H and K lines, $A_{k1} = 1.66 \times 10^8$ and $\overline{R}_k^2 = 23a_0^2$; for $\lambda 4227$, $A_{k1} = 1.40 \times 10^8$ and $\overline{R}_k^2 = 69a_0^2$.

In the computed variation of η through the atmosphere, it is interesting to note that, for the Ca II H and K lines, η remains practically constant in the regions relevant to the formation of the lines, decreasing with τ at greater depths. In the case of the $\lambda 4227$ line of Ca I, η first increases and then decreases through the atmosphere. In the wings of the sodium D lines, the variation of η with τ is similar to that for $\lambda 4227$, but the relative changes are less. At the centers of the D lines, η decreases with τ .

Contours have been computed at three points on the solar disc, at $\vartheta = 21^\circ, 48^\circ$, and 76° , according to the third approximation. The theoretical results are compared with the observational material presented by Houtgast.¹⁷ Tables 5–9 give the observed and computed values of the residual intensities at selected points on the contours as functions

¹⁵ *Zs. f. Ap.*, **21**, 254, 1942.

¹⁶ *Ap. J.*, **91**, 202, 1940.

¹⁷ "The Variations in the Profiles of Strong Fraunhofer Lines along a Radius of the Solar Disc" (dissertation), Utrecht, 1942.

¹⁸ *Ap. J.*, **102**, 385, 1945.

²⁰ *Phys. Zs. Sowjetunion*, **6**, 368, 1934.

¹⁹ *Zs. f. Phys.*, **72**, 697, 1931.

²¹ *Proc. R. Soc., A*, **164**, 167, 1938.

of ϑ . The observed contours at the given angles were obtained by linear interpolation in Houtgast's tables of residual intensities for undisturbed points. The observations for the sodium D lines are the most free from error, the calcium lines, especially λ 4227, being more strongly disturbed by blends. In the present work no proper theory of the central intensities is included. It is to be noted that the center-limb variations are most pronounced in the inner wings of the lines. The predicted and the observed profiles of the K line of Ca II, the Ca I line λ 4227, and the D₁ line of Na I at $\vartheta = 48^\circ$ are shown in Figures 1, 2, and 3, respectively. The agreement between theory and observations is satisfactory.

The quantities c , given by the equation

$$1 - r = \frac{c\,r}{\Delta\lambda^2}, \tag{97}$$

TABLE 5
RESIDUAL INTENSITIES FOR Ca II K

$\Delta\lambda$	$\vartheta = 21^\circ$		$\vartheta = 48^\circ$		$\vartheta = 76^\circ$	
	Computed	Observed	Computed	Observed	Computed	Observed
0.....	0	0.072	0	0.078	0	0.129
1.5.....	0.088	.150	0.105	.179	0.126	.270
3.....	.204	.241	.226	.285	.300	.397
5.....	.374	.395	.402	.421	.514	.531
7.....	.518	.558	.542	.562	.666	.647
10.....	.688	.725	.700	.724	.802	.767
15.....	.831	.849	.832	.848	.899	.868
20.....	0.895	0.924	0.896	0.920	0.940	0.936
c	47.7	38.8	46.8	38.0	25.9	32.4
Eq. widths.....	19,050	19,050	18,680	18,360	14,360	15,850

TABLE 6
RESIDUAL INTENSITIES FOR Ca II H

$\Delta\lambda$	$\vartheta = 21^\circ$		$\vartheta = 48^\circ$		$\vartheta = 76^\circ$	
	Computed	Observed	Computed	Observed	Computed	Observed
0.....	0	0.073	0	0.080	0	0.128
1.5.....	0.134	.193	0.150	.215	0.196	.320
3.....	.285	.325	.314	.353	.411	.469
5.....	.500	.500	.528	.523	.647	.616
7.....	.673	.655	.679	.660	.779	.731
10.....	.803	.791	.807	.799	.879	.842
15.....	.899	.891	.900	.894	.944	.928
20.....	0.941	0.932	0.941	0.936	0.967	0.958
c	24.3	26.0	23.9	24.8	13.5	18.1
Eq. widths.....	14,280	15,110	13,990	14,685	10,970	10,930

applicable to the wings, were found graphically for the computed contours. The equivalent widths were measured with a planimeter. The contributions to the equivalent widths from the far wings were obtained by the expression

$$\sqrt{c} \left(\frac{\pi}{2} - \tan^{-1} \frac{\Delta\lambda}{\sqrt{c}} \right). \quad (98)$$

The constants c and the equivalent widths of the lines, expressed in mÅ, are also given in Tables 5–9. The observed values were read from smooth curves drawn through Houtgast's observations across the solar disc for each line. In the ratios of the computed c 's for the $Ca\text{ II}$ H and K lines and for the $Na\text{ I}$ D lines, the doublet ratio 1:2 is reflected. In

TABLE 7
RESIDUAL INTENSITIES FOR $Ca\text{ I } \lambda 4227$

$\Delta\lambda$	$\vartheta = 21^\circ$		$\vartheta = 48^\circ$		$\vartheta = 76^\circ$	
	Computed	Observed	Computed	Observed	Computed	Observed
0.....	0.002	0.052	0.002	0.054	0.002	0.071
0.10.....	.066	.131	.080	.140	.11	.188
0.30.....	.280	.329	.303	.329	.398	.414
0.45.....	.466	.484	.479	.472	.598	.523
0.60.....	.609	.617	.624	.584	.725	.616
0.80.....	.740	.729	.735	.689	.817	.696
1.20.....	.867	.850	.858	.822	.902	.803
1.60.....	.923	.912	.916	.895	.942	.884
2.40.....	0.964	0.969	0.960	0.955	0.971	0.969
c	0.23	0.22	0.25	0.27	0.15	0.25
Eq. widths.....	1391	1425	1392	1519	1118	1449

TABLE 8
RESIDUAL INTENSITIES FOR $Na\text{ I } D_2$

$\Delta\lambda$	$\vartheta = 21^\circ$		$\vartheta = 48^\circ$		$\vartheta = 76^\circ$	
	Computed	Observed	Computed	Observed	Computed	Observed
0.....	0.004	0.070	0.006	0.060	0.006	0.085
0.068.....	.031	.111	.033	.111	.039	.111
0.12.....	.177	.218	.196	.229	.253	.235
0.20.....	.316	.370	.344	.392	.430	.450
0.40.....	.610	.639	.630	.640	.723	.698
0.60.....	.781	.793	.782	.784	.852	.814
0.80.....	.861	.878	.864	.866	.909	.880
1.20.....	.932	.934	.940	.935	.956	.947
1.60.....	.959	.949	.957	.959	.975	.970
2.00.....	0.971	0.968	0.973	0.970	0.983	0.979
c	0.10	0.092	0.10	0.097	0.060	0.087
Eq. widths.....	1003	912	984	926	773	878

the case of the observed ratios for the H and K lines the deviation from the doublet ratio is attributed to the disturbance of the H line by $H\epsilon$.

The ratios between the c 's and between the equivalent widths of a given line at various points on the solar disc are of greatest interest for comparison. These ratios, theoretical and observed, are recorded in Table 10. For the sodium D lines and λ 4227 of calcium the observations indicate an increase in the wings up to $\vartheta = 60^\circ$ and 65° , respectively. The small observed increase for the sodium D lines is not predicted. In the case of λ 4227 the

TABLE 9
RESIDUAL INTENSITIES FOR Na I D₁

$\Delta\lambda$	$\vartheta = 21^\circ$		$\vartheta = 48^\circ$		$\vartheta = 76^\circ$	
	Computed	Observed	Computed	Observed	Computed	Observed
0.....	0.008	0.105	0.008	0.100	0.008	0.105
0.068.....	.041	.165	.043	.160	.050	.155
0.12.....	.248	.325	.272	.326	.329	.340
0.20.....	.445	.500	.474	.511	.576	.558
0.40.....	.732	.771	.753	.755	.830	.795
0.60.....	.881	.884	.879	.880	.918	.883
0.80.....	.929	.925	.926	.923	.952	.918
1.20.....	.965	.965	.967	.965	.978	.969
1.60.....	.980	.987	.980	.984	.986	.989
2.00.....	0.988	0.995	0.989	0.995	0.992	0.990
c	0.051	0.047	0.050	0.050	0.032	0.043
Eq. widths.....	724	645	693	664	565	610

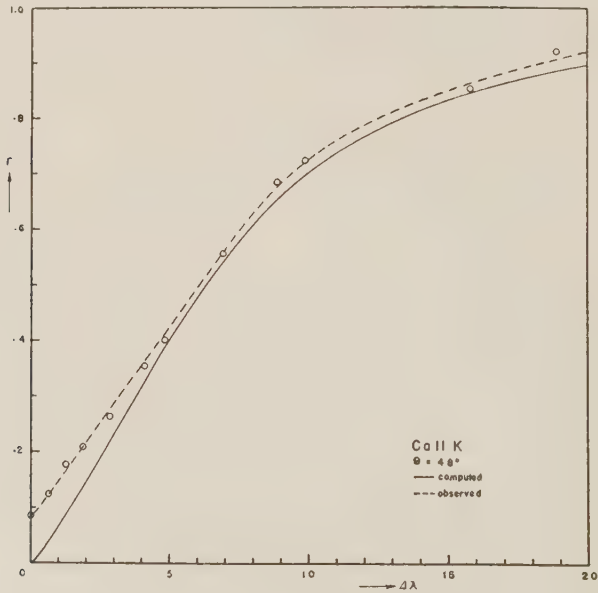


FIG. 1

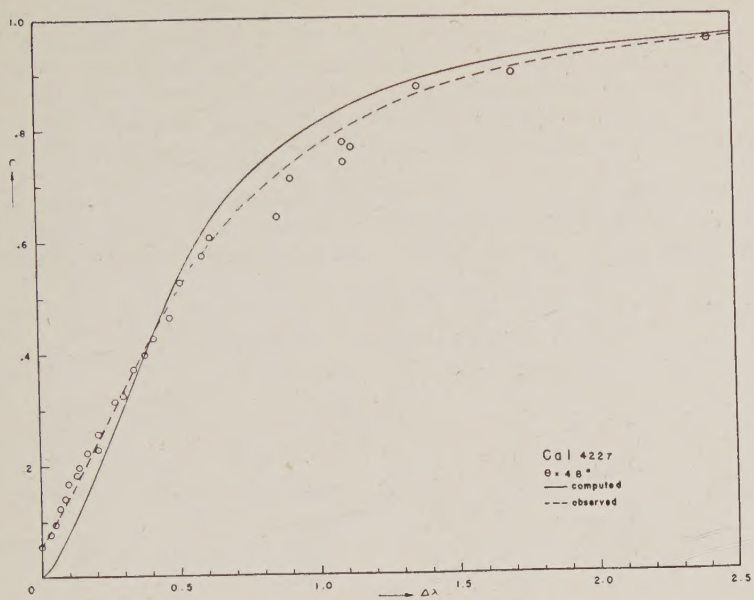


FIG. 2

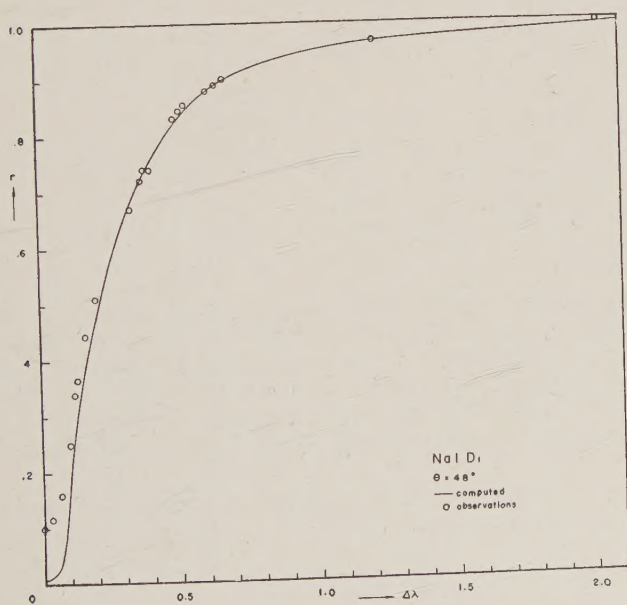


FIG. 3

predicted increase in the wings is less than the observed. This phenomenon is not so pronounced in the observations made by Royds and Narayan.²² In general, the theory predicts a greater decrease near the limb in the equivalent widths than is observed. Shane's²³ observations of the sodium D lines suggest, however, that the wings are appreciably weaker near the limb than Houtgast's data indicates. Also, for *Ca* I, λ 4227, Royds and Narayan²² find a decrease in the equivalent width at $\vartheta = 76^\circ$; the ratios of the equivalent widths obtained by them are 1.00, 1.02, and 0.904, the values of ϑ being 21° , 48° , and 76° , respectively.

In judging the differences which exist between the predicted and observed variations in the contours, it must be remembered that on the theoretical side the present work can be improved by the use of a more accurate model atmosphere for the sun. The Rosseland

TABLE 10
RATIOS OF THE ϵ 'S AND THE EQUIVALENT WIDTHS

LINE	$\frac{\epsilon(\vartheta)}{\epsilon(21^\circ)}$		$\frac{\text{Eq. Widths } (\vartheta)}{\text{Eq. Widths } (21^\circ)}$	
	Computed	Observed	Computed	Observed
<i>Ca</i> II K	$\vartheta = 21^\circ$	1	1	1
	$\vartheta = 48^\circ$	0.981	0.981	0.964
	$\vartheta = 76^\circ$	0.543	0.754	0.832
<i>Ca</i> II H	$\vartheta = 21^\circ$	1	1	1
	$\vartheta = 48^\circ$	0.984	0.980	0.972
	$\vartheta = 76^\circ$	0.556	0.768	0.723
<i>Ca</i> I λ 4227	$\vartheta = 21^\circ$	1	1	1
	$\vartheta = 48^\circ$	1.09	1.00	1.07
	$\vartheta = 76^\circ$	0.652	0.804	1.02
<i>Na</i> I D ₂	$\vartheta = 21^\circ$	1	1	1
	$\vartheta = 48^\circ$	1.00	0.981	1.02
	$\vartheta = 76^\circ$	0.600	0.771	0.963
<i>Na</i> I D ₁	$\vartheta = 21^\circ$	1	1	1
	$\vartheta = 48^\circ$	0.980	0.957	1.03
	$\vartheta = 76^\circ$	0.627	0.780	0.946

mean absorption coefficient used by Rudkjøbing should be replaced by the proper mean absorption coefficient as defined by Chandrasekhar.²⁴ Further, this should be determined with the new absorption curve for H^- ,²⁵ the contribution to the opacity from the free-free transitions of free electrons being included. On the basis of the model atmosphere obtained, the relative abundances of the various elements composing the solar atmosphere should be rederived. Such an investigation is proposed for a later time.

It is a pleasure to acknowledge my indebtedness to Dr. S. Chandrasekhar for his constant guidance and friendly encouragement.

²² *Kodaikanal Obs. Bull.*, No. 109, p. 375, 1936.

²³ *Lick Obs. Bull.*, No. 507, 1941.

²⁴ *A. J.*, 101, 328, 1945.

²⁵ S. Chandrasekhar, *A. J.*, 102, 395, 1945.

